Matrice, operacije sa matricama, determinanta i inv. matrice

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}_{m \times n}$$

$\rightarrow$ vrsta

$\uparrow$ kolona

a_{ij} - elem matrice na presjeku i-te vr i j-te kolone.

① Za date matrice ispitati da li se mogu izvršiti naznačene operacije, a onda ih izvesti:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 4 & 1 & -3 \\ 8 & 1 & -2 \end{bmatrix}$$

a) $3 \cdot A - B$; b) $3A + C$, c) $A - E$ d) $B + 4E$.

Rj. a) $\dim(A) = 2 \times 3$, $\dim B = 2 \times 2$, $\dim C = 2 \times 3$

nemoguće

b) $3A - C = \begin{bmatrix} 6 & 3 & -3 \\ 0 & 9 & -12 \end{bmatrix} - \begin{bmatrix} 4 & 1 & -3 \\ 8 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ -8 & 8 & -10 \end{bmatrix}_{2\times}$

c) $A - E$ → mora biti kvadratna ⇒ nemoguće.

d) $B + 4E = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ -1 & 7 \end{bmatrix}_{2\times 2}$

② Odrediti dimenzije matrice td. naznačena množenja budu moguća:

a) $(A)_{4\times m} \cdot (B)_{n\times 4} = (C)_{p\times q}$

$m = n \in \mathbb{N}$ $\quad p = 4,\ q = 4$

b) $(A)_{m\times 3} \cdot (B)_{n\times p} = (C)_{2\times 5}$

$m = 2,\ p = 5,\ n = 3$

c) $(A)_{4\times m} \cdot (B)_{2\times n} = (C)_{p\times 5}$

$m = 2,\ n = 5,\ p = 4$

d) $((A)_{4\times m} \cdot (B)_{n\times 3}) \cdot C_{p\times q} = D_{4\times 8}$

$m = n \in \mathbb{N}$

$p = 3$ $\quad q = 8$

③ Date su matrice: $A = \begin{pmatrix} 3 & 0 & -1 \\ 4 & 2 & 5 \end{pmatrix}$; $B = \begin{pmatrix} -1 & 2 \\ 4 & 5 \\ 7 & -1 \end{pmatrix}$, $C = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$

$D = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$.

Izračunati: a) $B \cdot C$, b) ABC; c) $A \cdot D$, d) $(2C - E) \cdot A$

e) $C^2 - 5 \cdot A \cdot B$ $\quad$ f) $C^T \cdot \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix} = \ldots$

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Rj.

a) $B \cdot C = \begin{pmatrix} -1 & 2 \\ 4 & 5 \\ 7 & -1 \end{pmatrix}_{3\times 2} \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}_{2\times 2} = \begin{pmatrix} 5 & 1 \\ 6 & 9 \\ -9 & 6 \end{pmatrix}$

b) $A \cdot B \cdot C = \begin{pmatrix} 3 & 0 & -1 \\ 4 & 2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 5 & 1 \\ 6 & 9 \\ -9 & 6 \end{pmatrix} = \begin{pmatrix} 24 & -3 \\ -13 & 52 \end{pmatrix}$

c) $A \cdot D = \begin{pmatrix} 3 & 0 & -1 \\ 4 & 2 & 5 \end{pmatrix}_{2\times 3} \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}_{3\times 1} = \begin{pmatrix} 10 \\ 9 \end{pmatrix}_{2\times 1}$

d) $(2C - E) \cdot A = \begin{pmatrix} -3 & 2 \\ 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 & -1 \\ 4 & 2 & 5 \end{pmatrix} = \begin{pmatrix} -1 & 4 & 13 \\ 16 & 2 & 1 \end{pmatrix}$

e) $C^2 - 2AB = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} - 2 \begin{pmatrix} 3 & 0 & -1 \\ 4 & 2 & 5 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 4 & 5 \\ 7 & -1 \end{pmatrix} =$

$= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} - 2 \begin{pmatrix} -10 & 7 \\ 39 & 13 \end{pmatrix} = \begin{pmatrix} 23 & -14 \\ -78 & -23 \end{pmatrix}$

4 Data je matrica a) $A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$, b) $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ -3 & 2 & -1 \end{bmatrix}$.

Izracunati $p(A)$ ako je $p(x) = 3x^3 - 2x^2 + 4x - 5$

Rj.

b) $p(A) = 3A^3 - 2A^2 + 4A - 5E$

$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ -3 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ -3 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 \\ -5 & 7 & -2 \\ 4 & 0 & 10 \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} 4 & -2 & 0 \\ -5 & 7 & -2 \\ 4 & 0 & 10 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ -3 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -7 \\ 15 & 3 & 28 \\ -26 & 20 & -14 \end{bmatrix}$

$$p(A) = \begin{pmatrix} 6 & -3 & -21 \\ 45 & 9 & 84 \\ -78 & 60 & -42 \end{pmatrix} - \begin{pmatrix} 8 & -2 & 0 \\ -10 & 14 & -4 \\ 8 & 0 & 20 \end{pmatrix} + \begin{pmatrix} 4 & 0 & -4 \\ 8 & 4 & 12 \\ -12 & 8 & -4 \end{pmatrix} - \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} =$$

$$= \begin{pmatrix} -3 & -1 & -25 \\ 63 & -6 & 100 \\ -98 & 68 & -71 \end{pmatrix}.$$

⑤ Dat je skup matrica $M(a) = \begin{pmatrix} 2-a & a-1 \\ 2(1-a) & 2a-1 \end{pmatrix}$.

a) Izračunati $M(a) \cdot M(b)$.

b) Odrediti a td $M(a) = E$

c) Izračunati $M^8(a)$, $n \in \mathbb{N}$

Rj

a) $M(a) M(b) = \ldots = \begin{pmatrix} 2-ab & ab-1 \\ 2(1-ab) & 2ab-1 \end{pmatrix} = M(ab)$

b) $2-a=1 \Rightarrow a=1$

$a-1=0 \quad a=1$

$2(1-a)=0 \quad a=1$

$2a-1=1 \quad a=1$

(T)

c) $M^8(a) = \underbrace{M(a)\, M(a) \cdots M(a)}_{8 \text{ puta}} = M(a \cdots a) = M(a^8) = \begin{pmatrix} 2-a^8 & a^8-1 \\ 2(1-a^8) & 2a^8-1 \end{pmatrix}.$

6 Riješiti matricne jednacine:

$$2X^T - B = 2A - E - 4X^T \text{ , gdje su}$$

$$A = \begin{pmatrix} -1 & 3 & 4 \\ 0 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix} \text{ i } B = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & -1 & 3 \end{pmatrix}.$$

Rj

$$6X^T = 2A - E + B \quad / \cdot \frac{1}{6}$$

$$X^T = \frac{1}{6}(2A - E + B)$$

$$\underline{X = \frac{1}{6}(2A - E + B)^T}$$

$$X = \frac{1}{6}\left(\begin{pmatrix} -2 & 6 & 8 \\ 0 & 4 & -2 \\ -2 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & -1 & 3 \end{pmatrix}\right)^T$$

$$X = \frac{1}{6}\begin{pmatrix} -1 & 5 & 8 \\ 0 & 3 & 0 \\ -1 & -1 & 4 \end{pmatrix}^T$$

$$\underline{X = \frac{1}{6}\begin{pmatrix} -1 & 0 & -1 \\ 5 & 3 & -1 \\ 8 & 0 & 4 \end{pmatrix}}$$

7 Riješiti matricnu jednacinu $2X - B^T = A - 3E - 7X$ gdje je $A = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 \\ -1 & 3 \end{bmatrix}$.

Determinanta i inverzna matrica

Samo za kvadratne matrice.

1 Izračunati det A;:

a) $|3| = 3$; $|-7| = -7$

b) $\begin{vmatrix} 4 & -1 \\ 2 & 8 \end{vmatrix} = 4 \cdot 8 - 2 \cdot (-1) = 32 + 2 = 24$

$\begin{vmatrix} -3 & 3 \\ 5 & 2 \end{vmatrix} = -3 \cdot 2 - 5 \cdot 3 = -6 - 15 = -21$

c) $\begin{vmatrix} 4 & -1 & 2 \\ 0 & 3 & -2 \\ -3 & 1 & 7 \end{vmatrix} = a_{11} A_{11} + a_{21} A_{21} + a_{31} A_{31} =$

$= 4 \cdot A_{11} + 0\, A_{21} + (-3) \cdot A_{31} = (*)$

$A_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 3 & -2 \\ 1 & 7 \end{vmatrix} = + \begin{vmatrix} 3 & -2 \\ 1 & 7 \end{vmatrix} = 21 + 2 = 23$

$A_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 2 \\ 1 & 7 \end{vmatrix} = - \begin{vmatrix} -1 & 2 \\ 1 & 7 \end{vmatrix} = -(-7 - 2) = 9$

$A_{31} = (-1)^{3+1} \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} = + \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} = 2 - 6 = -4$

$(*) = 4 \cdot 23 + 0 \cdot 9 - 3 \cdot (-4) = 92 + 12 = \underline{104}$

$\begin{vmatrix} 4 & -1 & 2 \\ 0 & 3 & -2 \\ -3 & 1 & 7 \end{vmatrix} = 4 \begin{vmatrix} 3 & -2 \\ 1 & 7 \end{vmatrix} + 1 \begin{vmatrix} 0 & -2 \\ -3 & 7 \end{vmatrix} + 2 \begin{vmatrix} 0 & 3 \\ -3 & 1 \end{vmatrix} = 4 \cdot 23 + 6 + 2 \cdot 9 =$

$= \underline{104}$

(F)

Sarrusovo pravilo (samo 3x3)

$$\begin{array}{ccc|cc} 4 & -1 & 2 & 4 & -1 \\ 0 & 3 & -2 & 0 & 3 \\ -3 & 1 & 7 & -3 & 1 \end{array} = 84 - 6 + 18 + 8 = 104$$

(2) Riješiti jednadžbe:

a) $\begin{vmatrix} 4-x & 1 \\ 3 & 2+x \end{vmatrix} = 3$ b) $\begin{vmatrix} -x-5 & 3 & 1 \\ -3 & 4-x & 1 \\ -3 & 3 & 1 \end{vmatrix} = 0$

Rj.

a) $(4-x)(2+x) - 3 = 3$

$8 + 2x - x^2 - 6 = 0$

$-x^2 + 2x + 2 = 0$

$x_{1,2} = \dfrac{-2 \pm \sqrt{4+8}}{-2}$

$x_{1,2} = \dfrac{-2 \pm 2\sqrt{3}}{-2}$ $\to x_1 = 1 - \sqrt{3}$

$\to x_2 = 1 + \sqrt{3}$

b) $(-x-5)(4-x-3) - 3\underbrace{(-3+3)}_{=0} + 1(-9+12$

$(-x-5)(1-x) + \underbrace{(3-3x)}_{3(1-x)} = 0$

$(1-x)(-x-5+3) = 0$

$(1-x)(-x-2) = 0$

$\underline{x = 1} \lor \underline{x = -2}$

③ Svodeći na trougaoni oblik izračunati:

a)
$$\begin{vmatrix} 2 & 1 & -1 \\ 3 & 4 & -2 \\ -2 & -3 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & -1 \\ 4 & 3 & -2 \\ -3 & -2 & 3 \end{vmatrix} \begin{matrix} (-4) \\ \cdot 3 \\ \end{matrix} =$$

$$= - \begin{vmatrix} 1 & 2 & -1 \\ 0 & -5 & 2 \\ 0 & 4 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -5 \\ 0 & 0 & 4 \end{vmatrix} = \boxed{8}$$

b)
$$\begin{vmatrix} 1 & 1 & 4 & 2 \\ 2 & -1 & 3 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & -1 & 0 & 1 \end{vmatrix} \begin{matrix} (-2) \\ (-1) \end{matrix} = \begin{vmatrix} 1 & 1 & 4 & 2 \\ 0 & -3 & -5 & -3 \\ 0 & 2 & 0 & 0 \\ 0 & -2 & -4 & -1 \end{vmatrix} = - \begin{vmatrix} 1 & 4 & 1 & 2 \\ 0 & -5 & -3 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & -4 & -2 & -1 \end{vmatrix} \oplus =$$

$$= - \begin{vmatrix} 1 & 4 & 1 & 2 \\ 0 & -5 & -3 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & \cancel{4} & 0 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 4 & 2 \\ 0 & -3 & -5 & -3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -4 & -1 \end{vmatrix} \begin{matrix} \cdot 2 \\ \cdot 3 \end{matrix} \oplus =$$

$$= \frac{1}{3} \begin{vmatrix} 1 & 1 & 4 & 2 \\ 0 & -3 & -5 & -3 \\ 0 & 0 & -10 & -6 \\ 0 & 0 & -4 & -1 \end{vmatrix} = -\frac{2}{3} \begin{vmatrix} 1 & 1 & 4 & 2 \\ 0 & -3 & -5 & -3 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & -4 & -1 \end{vmatrix} \begin{matrix} \cdot 4 \\ \cdot 5 \end{matrix} \oplus =$$

$$= -\frac{2}{3} \cdot \frac{1}{5} \begin{vmatrix} 1 & 1 & 4 & 2 \\ 0 & -3 & -5 & -3 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 7 \end{vmatrix} = \frac{-2}{\cancel{3} \cdot \cancel{5}} \cdot 1 \cdot \cancel{(-3)} \cdot \cancel{5} \cdot 7 = \boxed{14}$$

④ Naći karakteristični polinom matrice:

$$A = \begin{bmatrix} 4 & -1 & -2 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

Rj: $\det(A - \lambda E) = \text{polinom}$

$$\begin{vmatrix} 4-\lambda & -1 & -2 \\ 2 & 1-\lambda & -2 \\ 1 & -1 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda)^2 + 2 + 4 + 2(1-\lambda) - 2(4-\lambda) + 2(1-\lambda) =$$

$$= (4-\lambda)(1-2\lambda+\lambda^2) \cancel{+6+2} - 2\lambda \cancel{-8} + \cancel{2\lambda} + 2 - \cancel{2\lambda} =$$

$$= 4 - 8\lambda + 4\lambda^2 - \lambda + 2\lambda^2 - \lambda^3 - 2\lambda + 2 =$$

$$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6.$$

Primjer 16. Svesti na trougaoni oblik matricu $A=\begin{pmatrix}0 & -1 & -1 & -3\\ 1 & 2 & 4 & 7\\ 5 & 0 & 10 & 5\end{pmatrix}$ pomoću elementarnih transformacija.

Rješenje. $A=\begin{pmatrix}0 & -1 & -1 & -3\\ 1 & 2 & 4 & 7\\ 5 & 0 & 10 & 5\end{pmatrix} \mathrm{I}\leftrightarrow \mathrm{II} \sim \begin{pmatrix}1 & 2 & 4 & 7\\ 0 & -1 & -1 & -3\\ 5 & 0 & 10 & 5\end{pmatrix} \mathrm{III}-5\cdot \mathrm{I} \sim$

$\begin{pmatrix}1 & 2 & 4 & 7\\ 0 & -1 & -1 & -3\\ 0 & -10 & -10 & -30\end{pmatrix} \mathrm{III}-10\cdot \mathrm{II} \sim B=\begin{pmatrix}1 & 2 & 4 & 7\\ 0 & -1 & -1 & -3\\ 0 & 0 & 0 & 0\end{pmatrix}$. Matrica B je trougaonog oblika i A~B.

Zadaci za vježbu

1. Naći matricu:

a) $2\cdot A+3\cdot B$, ako je $A=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & -1\end{pmatrix}$ i $B=\begin{pmatrix}-2 & 3 & 0\\ 2 & 1 & 1\end{pmatrix}$,

b) $4\cdot A-5\cdot B$, ako je $A=\begin{pmatrix}2 & -1 & 0\\ 3 & 4 & 1\\ -2 & 1 & 5\end{pmatrix}$ i $B=\begin{pmatrix}3 & 1 & 2\\ -2 & 9 & 0\\ 3 & 4 & -1\end{pmatrix}$.

Rješenje. a) $\begin{pmatrix}-4 & 13 & 6\\ 6 & 5 & 1\end{pmatrix}$, b) $\begin{pmatrix}-7 & -9 & -10\\ 22 & -29 & 4\\ -23 & -16 & 25\end{pmatrix}$.

2. Naći proizvod matrica $A\cdot B$ i $B\cdot A$ (ako postoji):

a) $A=\begin{pmatrix}1 & 3\\ 2 & -4\end{pmatrix}$ i $B=\begin{pmatrix}3 & -2\\ 6 & 1\end{pmatrix}$; b) $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ i $B=\begin{pmatrix}1 & 3\\ -2 & 2\\ 0 & 1\end{pmatrix}$;

c) $\begin{pmatrix}1 & 3 & 2\\ -2 & 4 & -3\end{pmatrix}$ i $B=\begin{pmatrix}1 & -2 & 0\\ 5 & 2 & 6\\ 0 & -3 & 3\end{pmatrix}$.

Rješenje. a) $A\cdot B=\begin{pmatrix}21 & 1\\ -18 & -8\end{pmatrix}$, $B\cdot A=\begin{pmatrix}-1 & 17\\ 8 & 14\end{pmatrix}$. b) $A\cdot B$ ne postoji, $B\cdot A=\begin{pmatrix}10 & 14\\ 4 & 4\\ 3 & 4\end{pmatrix}$.

c) $A\cdot B=\begin{pmatrix}16 & -2 & 24\\ 18 & 21 & 15\end{pmatrix}$, $B\cdot A$ ne postoji.

3. Naći vrijednost matričnog polinoma f(A), ako je $f(x)=-2x^2+3x-5$ i $A=\begin{pmatrix}2 & 3\\ 1 & -1\end{pmatrix}$.

Rješenje. $\begin{pmatrix}-13 & 3\\ 1 & -16\end{pmatrix}$.

4. Naći AA^T i A^TA, ako je $A=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}$.

Rješenje. $AA^T=\begin{pmatrix}14 & 32 & 50\\ 32 & 77 & 122\\ 50 & 32 & 194\end{pmatrix}$, $A^TA=\begin{pmatrix}66 & 78 & 90\\ 78 & 93 & 108\\ 90 & 108 & 126\end{pmatrix}$.

5. Svesti na trougaoni oblik matricu: a) $A=\begin{pmatrix}2 & 3 & -2\\ 3 & 1 & 1\\ 1 & 5 & -5\end{pmatrix}$, b) $A=\begin{pmatrix}1 & 2 & -1 & 0\\ 3 & -1 & 2 & 2\\ 2 & 5 & -1 & 0\\ 1 & -1 & 0 & 2\end{pmatrix}$.

Rješenje. a) $\begin{pmatrix}2 & 3 & -2\\ 0 & -7 & 8\\ 0 & 0 & 0\end{pmatrix}$, b) $\begin{pmatrix}1 & 2 & -1 & 0\\ 0 & -7 & 5 & 2\\ 0 & 0 & 12 & 2\\ 0 & 0 & 0 & 14\end{pmatrix}$.

7. Odgovori na pitanja:

a) Ako se matrice A i B mogu množiti da li to znači da se one mogu sabirati? (Ne).

b) Ako se matrice A i B mogu sabirati da li to znači da se one mogu množiti? (Ne)

c) Mogu li matice A i A^T mogu biti jednake? (Da).

8. Za matricu $A=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$ naći A^2.

Rješenje. $A^2=\begin{pmatrix}\cos 2\alpha & -\sin 2\alpha\\ \sin 2\alpha & \cos 2\alpha\end{pmatrix}$.